AMATH 505 Final Project (Due online via Canvas: Dec 13th)

Team Member: Guang Hua, Avinash Joshi, Alireza Zarei, Ryan Chiu
Project Topic: Modeling 2D Turbulence (Numerical)

I. Introduction

We choose to analyze and simulate 2D fluid turbulence via pseudo-spectral method. 2D turbulence are observed in energized fluids dynamics, along with chaotic fluid motions over a wide range of spectrum length scales. In geophysics, we could observe these scenarios every single second in both global and local scale, which affects the development of weather system and ocean dynamics. Cyclones, for example, tends to stay stationary under low energy states, while under high energy states cyclones tends to move around and interact with other cyclones/anticyclones, either form new cyclone or dissipate by others. Analyzing 2D fluid turbulence could give us capability to current motion tendency of the vortices and give an approximation of the trajectory where it would go in the next time-frame.

There is a concept involved called inverse energy cascade, which means the energy transfer from small scale to larger scale. The transfer requires that the dynamical system has to be nonlinear, and this energy transfer gives an explanation of the transition from small vortices to large vortices in 2D turbulence system over the time. This concept was first introduced and predicted in the 1960s by Kraichnan, and then observed in a vast of scenarios, from small as vortices on the soap films to large as Jupiter's atmosphere. The result combined 2D turbulence and inverse energy cascade allows for a precise preparation of the initial state and measurement of both compressible and incompressible fluids.

II. Theoretical and algorithm

We first describe the mathematical formulation behind this problem. This mathematical formulation concept is based on Marin Lauber's report for 2D Turbulence simulation. We first start with using Helmholtz's vorticity equation to describe the transport of vorticity by applying curl operator to the Navier-Stokes formula:

$$\nabla \times \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u} \right)$$

Here $\vec{u}$ is the velocity vector field of the fluid parcels, and p is the pressure field. For 2D situations we could simplify the equation by appling following steps:

(a) Using definition of vorticity, we have $omega = \nabla \times \vec{u}$. And rearrange the terms:

$$\nabla \times \left(\frac{\partial \vec{u}}{\partial t} + \nabla \left(\frac{\vec{u}^2}{2} \right) + \omega \times \vec{u} \right) = \nabla \times \left(-\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u} \right)$$

$$\frac{\partial \omega}{\partial t} + \nabla \times \nabla \left(\frac{\vec{u}^2}{2} \right) + \nabla \times (\omega \times \vec{u}) = -\nabla \times \frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u}$$

$$\frac{\partial \omega}{\partial t} + \nabla \times \nabla \left(\frac{\vec{u}^2}{2} \right) + \vec{u} \cdot \nabla \omega - \vec{u} \nabla \cdot \omega - \omega \cdot \nabla \vec{u} + \omega \nabla \cdot \vec{u} = -\nabla \times \frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u}$$

(b) Identify zero terms:

- $\nabla \times \nabla \left(\frac{\vec{u}^2}{2}\right)$ is a curl of a tensor, which vanishes.
- $\vec{u}\nabla \cdot \omega$ is zero because $\nabla \cdot \omega = 0$.
- $\omega \nabla \cdot \vec{u} = 0$ for incompressible flow.
- $\nabla \times \frac{1}{\rho}\nabla p$ is zero as we could check via product rule:

$$\nabla \times \frac{1}{\rho}\nabla p = \nabla \left(\frac{1}{\rho}\right) \times \nabla p + \frac{1}{\rho}\nabla \times \nabla p = \frac{1}{\rho^2}\nabla \rho \times \nabla p$$

Here $\frac{1}{\rho^2}\nabla \rho \times \nabla p = 0$ for a flow with constant entropy.

Then we have the simplified version:

$$\frac{\partial \omega}{\partial t} + \vec{u} \cdot \nabla \omega = \nu \nabla^2 \omega$$

In 2D situation, note that angular velocity is conserved, and the length of a vortex tube cannot change due to continuity.

Now we need to find the stream function. We introduce a new function ψ. By definition of the stream function, we can write:

$$\vec{u} = \nabla \times \psi, \quad \vec{u}_i = \varepsilon_{ijk}\frac{\partial \psi_k}{\partial x_j}$$

Since we consider 2D plane only, so the only nonzero components of the velocity fields are:

$$\vec{u}_1 = \varepsilon_{123}\frac{\partial \psi_3}{\partial x_2} = \frac{\partial \psi_3}{\partial x_2}$$

$$\vec{u}_2 = \varepsilon_{213}\frac{\partial \psi_3}{\partial x_1} = -\frac{\partial \psi_3}{\partial x_1}$$

We then could obtain the velocity component in partial derivative form involving ψ function:

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$

We could use the equation of the transport of vorticity to find ω in term of ψ by using $\omega = \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$:

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{1}{\text{Re}}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)$$

$$\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = \frac{1}{\text{Re}}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)$$

$$-\omega = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

In algorithm wise, we choose to use third-order Runge-Kutta TVD (*Total Variance Diminishing*) method. This is an implicit method, therefore we need to approximate the time step as the following:

$$
\begin{aligned}
\tilde{\omega}^{(1)} &= \tilde{\omega}^{n} + \Delta t \mathcal{L}(\tilde{\omega}^{n}) \\
\tilde{\omega}^{(2)} &= \frac{3}{4}\tilde{\omega}^{n} + \frac{1}{4}\tilde{\omega}^{(1)} + \frac{1}{4}\Delta t \mathcal{L}(\tilde{\omega}^{(1)}) \\
\tilde{\omega}^{(n+1)} &= \frac{1}{3}\tilde{\omega}^{n} + \frac{2}{3}\tilde{\omega}^{(2)} + \frac{2}{3}\Delta t \mathcal{L}(\tilde{\omega}^{(2)})
\end{aligned}
$$

Where $\mathcal{L}(\tilde{\omega}_n)$ denotes all the spatial operators of the transport equation of the Fourier coefficients.

We use Taylor-Green vortex formula to validate our results. This is a known solution to the Navier-Stokes Equation and is stated follows:

$$\omega^e(x, y, t) = 2\kappa \cos(\kappa x)\cos(\kappa y) e^{-2\kappa^2 t/\mathrm{Re}}$$

Where κ is the number of vortices in each direction and Re is the Reynolds Number of the flow. We will use this formula to check the error and see whether our solution converges or not.

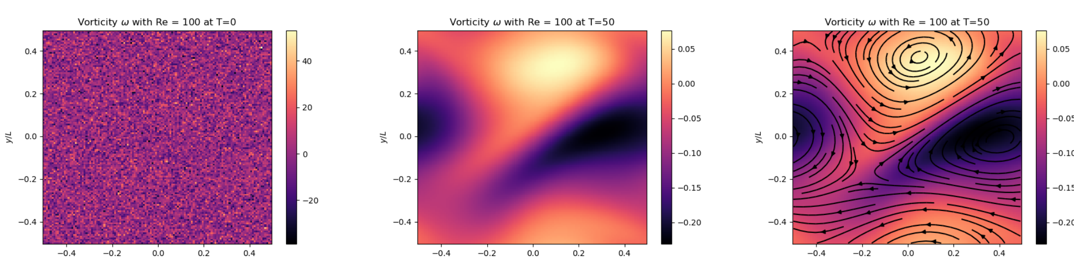
To show the trend of the Energy Spectrum, we use the following formula:

$$E(k) = \sum_{k \le \mathbf{k} \le k+dk} \frac{1}{2} k^2 |\hat{\omega}|^2(k).$$

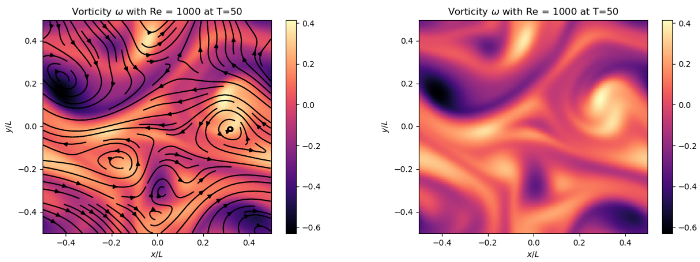
III. Result and Visualization

We put these all together, and made it run in the code. The python code will initialize the property of each vortices requested, including its vorticity, initial position, Reynolds number, spectral length for the grid, etc. Then the code will compute the Fourier transform for the ψ function, then use the result to compute the convection for the vortices. Combined with diffusion term, we could generate the velocity trajectory for each vortices on the grid and make a short animation across the time span. We will normalize the result to get a better visual representation. Here is some result graphs we obtained:

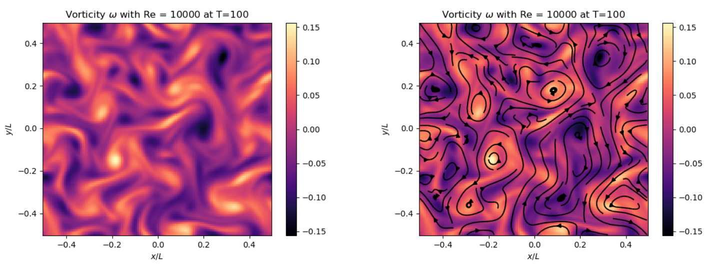
- Re = 100, N = 128, T = [0,50]:



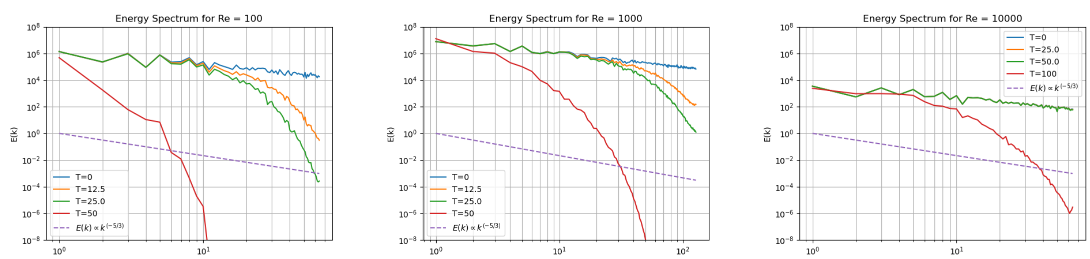
- Re = 1000, N = 256, T = [0,50]:



- Re = 10000, N = 128, T = [0,100]



We could observe that the vortices rotates and combined into larger vortices over the time, and diffuses out. In addition, from the plot generated, as Reynolds number increases, the flow becomes more turbulent, therefore within the same time span there are more small vortices exists for higher Reynolds number flow than lower Reynolds number flow. We could also see the energy spectrum generated from the above cases:



Notice there is a upward shift in the graph compare to the ideal trend fit. This is because we did not compute the exact value of ε. But overall we could observe that the energy at $T = 0$ is proportional to the $k^{-5/3}$, which is actually the "enstrophy" of the fluid. As time goes on, the energy trend is more towards to be proportional to k^{-4}, which is the actual energy decay of the system.

IV. Comparison and Conclusion

Finally we check the L2-norm error compare to the Taylor-Green vortex result, with the setting to Re = 100, N = 128, T = [0,50]:

```
_ = test_sim(dt = 10**-2, T = 50, Re = 100, N = 128, kappa = 4)
```

1m 49.4s Python

```
Step 0 || sim time 0.0 s || iteration time 0.147178 s/it
Step 500 || sim time 5.0 s || iteration time 0.021102 s/it
Step 1000 || sim time 10.0 s || iteration time 0.024345 s/it
Step 1500 || sim time 15.0 s || iteration time 0.025992 s/it
Step 2000 || sim time 20.0 s || iteration time 0.025277 s/it
Step 2500 || sim time 25.0 s || iteration time 0.024089 s/it
Step 3000 || sim time 30.0 s || iteration time 0.023340 s/it
Step 3500 || sim time 35.0 s || iteration time 0.022818 s/it
Step 4000 || sim time 40.0 s || iteration time 0.022408 s/it
Step 4500 || sim time 45.0 s || iteration time 0.022091 s/it
L2 norm error: 9.858811205210147e-15
```

Therefore the simulation is very accurate to the Taylor-Green's theoretical result.

V. Reference

- Siegelman, Lia, and William R. Young. "Two-dimensional turbulence above topography: Vortices and potential vorticity homogenization." Proceedings of the National Academy of Sciences, vol. 120, no. 44, 2023, https://doi.org/10.1073/pnas.2308018120.
- Marin Lauber. "Pseudo-Spectral Method for 2D Turbulence." Marin Lauber, 19 Nov. 2020, marinlauber.github.io/2D-Turbulence/.
- Omer San, et al. "High-Order Methods for Decaying Two-Dimensional Homogeneous Isotropic Turbulence." Computers and amp; Fluids, Pergamon, 21 Apr. 2012, www.sciencedirect.com/science/article/pii/S0045793012001363#s0015.
- University of California, Los Angeles, people.atmos.ucla.edu/jcm/turbulence_course_notes/2D_homogeneous.pdf. Accessed 3 Dec. 2023.
- Physics Courses, courses.physics.ucsd.edu/2019/Winter/physics116_216/annurev-fluid-120710-101240.pdf. Accessed 3 Dec. 2023.
- Durran, Dale, et al. "Practical Considerations for Computing Dimensional Spectra from Gridded Data." AMETSOC, American Meteorological Society, 1 Sept. 2017, journals.ametsoc.org/view/journals/mwre/145/9/mwr-d-17-0056.1.xml.